A Sequence of Families Converging in an Equiconvergent Manner

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The theorem of Arzela-Ascoli states that a necessary and sufficient condition for a family B of real-valued continuous functions defined on a compact metric space X to be compact in $C(X)$ is that the family B be uniformly bounded and equicontinuous. In showing that B is compact in $C(X)$ the difficult part generally is to show that B is equicontinuous. The following theorem may be of some value in this regard. We first, however, state a definition.

Definition: If the pair (X,d) is a metric space, we say that a sequence of families of real-valued functions $F_n = \{f_\alpha^n\}$, α belonging to some index set A, each $\{f_\alpha^n\}$ defined on X, converges to the family $\{f_\alpha\}$ in an *equiconvergent manner* as $n \to \infty$ if, for each $\varepsilon > 0$, there exists an integer N, independent of $\alpha \in A$, such that $n \geq N$ implies

$$\left| f_\alpha^n(x) - f_\alpha(x) \right| < \varepsilon$$

for all $x \in X$ and *all* $\alpha \in A$.

Theorem: If for each $n = 1, 2, \ldots,$ $F_n = \{f_\alpha^n\}$ is a family of equicontinuous real-valued functions defined on a compact metric space (X,d) and if $\{f_\alpha^n\}$ converges to $\{f_\alpha\}$ in an equiconvergent manner as $n \to \infty$, then the family $\{f_\alpha\}$ is equicontinuous.

Proof: Since each family F_n is equicontinuous, we know that for every $\varepsilon > 0$ there exists a $\delta_n > 0$, independent of α, such that

$$d(x,y) < \delta_n \;\Rightarrow\; \left| f_\alpha^n(x) - f_\alpha^n(y) \right| < \frac{\varepsilon}{3}$$

for all $\alpha \in A$ from the definition. Also since $\{f_\alpha^n\}$ converges to $\{f_\alpha\}$ in an *equiconvergent manner* we know that if $\varepsilon > 0$ then there exists an integer N independent of α such that for all $x \in X, \alpha \in A$,

$$\left| f_\alpha^n(x) - f_\alpha(x) \right| < \frac{\varepsilon}{3}$$

for all $n \geq N$. Now pick $\varepsilon > 0$. We conclude that there exists a $\delta_n > 0$, independent of $\alpha \in A$, such that if $d(x, y) < \delta_n$, then for all $\alpha \in A$ we have

$$\begin{aligned} \left| f_\alpha(x) - f_\alpha(y) \right| &\leq \left| f_\alpha(x) - f_\alpha^n(x) \right| + \left| f_\alpha^n(x) - f_\alpha^n(y) \right| + \left| f_\alpha^n(y) - f_\alpha(y) \right| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

This completes the proof.