Periodic Solutions of Nonlinear Boundary-Value Problems of the Second Kind

Abstract: We study the parabolic boundary-value problem of the second kind (which we call Problem A)

PDE: $Lu = f(x,t)$ $\qquad (x,t) \in D \times (-\infty, \infty)$

BC: $\dfrac{\partial u}{\partial \eta} + \beta u = g(x,t,u)$ $\qquad (x,t) \in \partial D \times (-\infty, \infty)$

where

1. $L = \sum \sum a_{ij}(x,t)\dfrac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(x,t)\dfrac{\partial}{\partial x_i} + c(x,t) - \dfrac{\partial}{\partial t}$ is a uniformly parabolic operator on $\bar{D} \times (-\infty, \infty)$

2. D is a smooth bounded domain in E^n.

3. The coefficients of L are continuous and satisfy the following Holder conditions on $\partial D \times (-\infty, \infty)$

$$\left|a_{ij}(x,t) - a_{ij}(x^0,t)\right| \le M \left|x - x^0\right|^\alpha$$
$$\left|b_i(x,t) - b_i(x^0,t)\right| \le M \left|x - x^0\right|^\alpha$$
$$\left|c\ (x,t) - c\ (x^0,t)\right| \le M \left|x - x^0\right|^\alpha$$

4. ∂D belongs to $C_{1+\alpha}$

5. f satisfies the Holder condition on $D \times (-\infty, \infty)$

$$\left|f(x,t) - f(x^0,t)\right| \le \left|x - x^0\right|^\alpha$$

and is uniformly continuous in x and t on $\partial D \times (-\infty, \infty)$.

6. β, g are continuous on $\partial D \times (-\infty, \infty)$

Theorem: For the above problem (which ensures the existence of at least one solution) there exists at least one solution $u = u(x,t)$ *periodic* in t with period T provided:

1. the functions $a_{ij}, b_i, c, f, \beta, g$ are periodic in t with period T
2. $\beta(x,t) \le \beta_0 < 0$, β_0 a constant on $(x,t) \in \partial D \times (-\infty, \infty)$

3. $c(x,t) \le 0,\ (x,t) \in \partial D \times (-\infty, \infty)$
4. $g = g(x,t,v)$ is continuous on $(x,t) \in \partial D \times (-\infty, \infty)$

Proof: The proof uses the Schauder fixed point theorem to show that the operator $S : v \to u$ defined by

$$Lu = f(x,t)$$
$$\frac{\partial u}{\partial \eta} + \beta(x,t)u = g(x,t,v)$$

has a fixed point in the normed space of continuous periodic functions functions on $(x,t) \in \overline{D} \times (-\infty, \infty)$.

Theorem 2: If $u = u(x,t)$ is a bounded solution of Problem A that satisfies the conditions of Theorem 1, and if $g(x,t,v)$ is monotone increasing in v then the solution $u = u(x,t)$ is unique.