Section 1.2: Conditional and Biconditional Connectives

Purpose of Lesson: To introduce the **conditional** and **biconditional** connectives and various equivalent forms.

The Conditional Sentence

The most important way to combine sentences in mathematics is the **conditional sentence** (or **implication**), which has the form "if *P* then *Q*." In a purely logical sense, conditional sentences do not necessarily imply a cause and effect between the components *P* and *Q*, although in mathematics and in general discourse they do. From a logical point of view the sentence

$$\textit{If } 1+1=3 \textit{ then pigs can fly}$$

is a legitimate implication, although there is no relationship between the component parts. On the other hand, when we write

If N is an integer, then 2N is an even integer

there is a definite cause and effect between the components. The reader has seen conditional sentences in Euclidean geometry where much of the subject is explained through implications. The sentence "If a polygon has three sides, then it is a triangle," is a conditional sentence.

Conditional Sentence : If *P* and *Q* are sentences, then the **conditional sentence** "if *P* then *Q*" is denoted symbolically by

$$P \Rightarrow Q$$

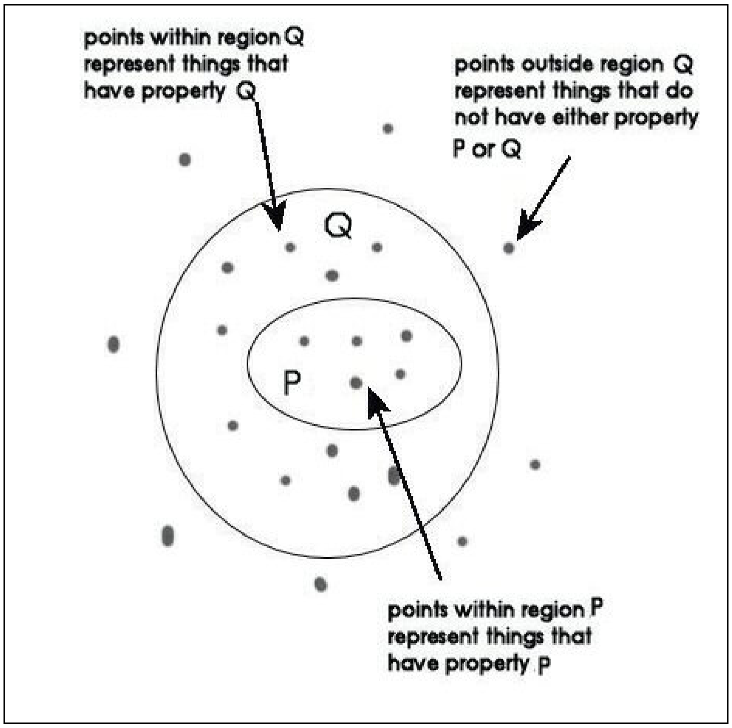
and the truth values of the sentence are defined by the truth table:

P	Q	$P \Rightarrow Q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

The sentence *P* is called the **assumption** (or **premise** or **antecedent**) of the implication and *Q* is called the **conclusion** (or **consequent**[1]).

[1] In pure logical systems *P* and *Q* are generally called the antecedent and consequent. In mathematics they are more likely to be called the assumption and conclusion.

The conditional statement $P \Rightarrow Q$ can be visualized by the Euler (or Venn) diagram as shown in Figure 1.



Euler Diagram for $P \Rightarrow Q$
Figure 1

Example 1: Conditional Sentences

The following sentences are conditional sentences.

i) If f is a real-valued differentiable function on $(-\infty, \infty)$, then f is continuous on $(-\infty, \infty)$.

ii) If N is an even number greater than 2, then N is the sum of two primes. (You get an A for the course if you can prove this.)

iii) If a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse, then $c^2 = a^2 + b^2$.

Understanding the Conditional Sentence: The conditional sentence "if P then Q" is best understood as a promise, where if the promise is kept, the conditional sentence is true, otherwise the sentence is false. As an illustration suppose your professor makes you the promise:

If pigs fly, then you will receive an A for the course.

Think about this for a second. If pigs really do fly and your professor gives you an A, your professor has kept his or her promise and the conditional sentence "if ⋯. then" is true. But, suppose pigs fly but your professor reneges and you do not get an A. Then your professor has broken the bond and the sentence "if ⋯ then" is false.

Now (here is where it gets confusing) suppose pigs don't fly, then what should your professor do? In this case the professor can do *anything* he or she so desires and still promise is kept, the argument being that the sentence "if ⋯ then" is true since the professor only promised an A if in fact pigs fly[2]. This line of reasoning jives with the truth table for the conditional sentence.

The conditional sentence $P \Rightarrow Q$ is sometimes called an **inference**, and we say that P **implies** Q. Another way of stating $P \Rightarrow Q$ in English is to say that P is a **sufficient condition** for Q, which means the truth of P guarantees the truth of Q. We can also say Q is a **necessary condition** for P, meaning that Q necessarily follows from P.

Example 2: Necessary Conditions and Sufficient Conditions.

P	Q	Condition
being pregnant	being female	Q is necessary for P
N is an integer	$2N$ is an integer	P is sufficient for Q
life on earth	air	Q is necessary for P
Getting run over by a steam roller	squashed	P is sufficient for Q

Necessary Conditions and Sufficient Conditions
Table 1

The sentence $P \Leftarrow Q$ (more often written $Q \Rightarrow P$) is called the **converse** of $P \Rightarrow Q$. A conditional and its converse are *not* logically equivalent. You could determine this from Table 2.

[2] Some people might argue that if pigs don't fly and the professor gives the student an A, then the sentence "if ⋯ then" should be considered false.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Implication and Converse
Table 2

Martin Note: In mathematics when one writes the implication $P \Rightarrow Q$ one generally that the sentence is true since we have assumed a true assumption P and have proven that the conclusion Q is true. In formal logic however, one allows for the possibility that P may be either true or false.

Biconditional

Theorems of the form "P if and only if Q" are highly valued in mathematics, giving **equivalent and interesting new ways to say exactly the same thing.** You have seen sentences of this type in Euclidean geometry. Do you remember the theorem: *A triangle is isosceles if and only if it has two congruent (equal) sides*?

This leads us to the following definition.

Definition: If P and Q are sentences, then the **biconditional** sentence " P if and only if Q" is denoted by

$$P \Leftrightarrow Q$$

whose truth values are given by the truth table

P	Q	$P \Leftrightarrow Q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

$P \Leftrightarrow Q$ is often read as "P **if and only if** Q" or P iff Q for shorthand. Another phrasing of $P \Leftrightarrow Q$ is P is a **necessary and sufficient condition** for Q.

Margin Note: In computer science and engineering the truth values of sentences are often 1 and 0 in place of T and F. Hence, the truth table for the biconditional is

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Example 3: Biconditional Equivalent to two Implications

Show that the biconditional $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Solution The truth values in the truth table under (5) and (6) are the same as seen in Table 3.

		(1)	(2)	(3)	(4)	(5)	(6)
P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	$P \Leftrightarrow Q$
T	T	F	F	T	T	**T**	**T**
T	F	F	T	F	T	**F**	**F**
F	T	T	F	T	F	**F**	**F**
F	F	T	T	T	T	**T**	**T**

Table 3

Warning: Be careful not to confuse the biconditional connective $P \Leftrightarrow Q$ with $P \equiv Q$ which says P and Q are *logical equivalent compound* sentences. The biconditional $P \Leftrightarrow Q$ does *not* necessarily mean P and Q have the same truth values (since the biconditional can be false), whereas $P \equiv Q$ does.

Margin Note: British mathematicians George Boole (1815-1664) and Augustus DeMorgan (1806-1871) started a renascence of logic in the mid-1800s. De Morgan's major contributions to logic include De Morgan's laws. George Boole approached logic in a new way, reducing it to simple algebra, replacing the truth value T by 1 and the truth value F by 0. Logical "and" becomes multiplication and logical "or" becomes addition in this new system. The result was Boolean algebra, the mathematical foundation for much of digital computers.

Example 4: Truth Values of Biconditional Sentences

Biconditional	**Truth Value**
$1+3=5$ if and only if $3+1=5$	True
The moon is round iff the earth is round	True
$1+2=3$ if and only if $\frac{\mathrm{d}}{\mathrm{dt}} e^t = te^t$	False
A necessary and sufficient condition for $1=0$ is $6/3=2$	False

In mathematics, the biconditional is stated in a variety of ways.

Equivalent Forms of the Biconditional

The following are examples of the biconditional sentence.

Equivalent Biconditional Forms	Example
$P \Leftrightarrow Q$	$x-1=0$ iff $x=1$
P if and only if Q	$x-1=0$ if and only if $x=1$
P iff Q	$x-1=0$ iff $x=1$
If P then Q and conversely	if $x-1=0$ then $x=1$ and conversely
If Q then P and conversely	If $x=1$ then $x-1=0$ and conversely
P is a necessary and sufficient condition for Q	$x-1=0$ is a necessary and sufficient condition for $x=1$
Q is a necessary and sufficient condition for P	$x=1$ is a necessary and sufficient condition for $x-1=0$

Example 5 Biconditional proposition in Differential Equations

Biconditional sentences allow one to replace one mathematical fact with another. For example, we solve the first-order linear differential equation by using a series of "if and only if" statements.

$$
\begin{aligned}
\frac{dy}{dt} + ay = 0 &\Leftrightarrow e^{at}\left[\frac{dy}{dt} + ay\right] = 0 \\
&\Leftrightarrow \frac{d\left(e^{at}y(t)\right)}{dt} = 0 \\
&\Leftrightarrow e^{at}y(t) = c \\
&\Leftrightarrow y(t) = c\,e^{-at}
\end{aligned}
$$

Problems

Working Definitions: The following definitions are needed in some problems in this and if following sections.

- An integer n **divides** an integer m (and we write $n|m$) if there exists an integer q such that $m = n \times q$.
- An integer n is even if there exists an integer k such that $n = 2k$.
- An integer n is odd if there exists an integer k such that $n = 2k + 1$.
- A natural number p is prime if it is only divisible by 1 and itself.

1. Identify the assumption and conclusion in the following conditional sentences and tell if the implication is true or false.

   a) If pigs fly then I am richer than Bill Gates.
   b) If a person got the plague in the 17^{th} century they were in trouble.
   c) If you miss class over 75% of the time you are in trouble.
   d) If x is a prime number then x^2 is prime too.
   e) If x and y are prime numbers, then so is $x + y$.
   f) If the determinant of a matrix is nonzero, the matrix has an inverse.
   g) If f is a 1-1 function then f has an inverse.

2. Write the converse and contrapositive of the conditional sentences in Problem 1.

3. Let P be the sentence "$4 > 6$", Q the sentence "$1+1=2$", and R the sentence "$1+1=3$". What is the truth value of the following sentences?

   a) $P \wedge \sim Q$
   b) $\sim(P \wedge Q)$
   c) $\sim(P \vee Q)$
   d) $\sim P \wedge \sim Q$
   e) $P \wedge Q$
   f) $P \Rightarrow Q$
   g) $Q \Leftrightarrow R$
   h) $P \Rightarrow (Q \Rightarrow R)$
   i) $(P \Rightarrow Q) \Rightarrow R$
   j) $(R \vee Q \vee R) \Leftrightarrow (P \wedge QR)$

4. Let P be the sentence "Jerry is richer than Mary", Q is the sentence "Jerry is taller than Mary", and R is the sentence "Mary is taller than Jerry." For the following sentences what can you conclude about Jerry and Mary if the sentences are true. Express the information in a convenient form.

a) $P \vee Q$
b) $P \wedge Q$
c) $\sim P \vee Q$
d) $Q \wedge R$
e) $\sim Q \wedge \sim R$
f) $P \wedge (P \Rightarrow Q)$
g) $P \Leftrightarrow (Q \vee R)$
h) $Q \wedge (P \Rightarrow R)$
i) $P \vee Q \vee R$
j) $P \vee (Q \wedge R)$

5. Construct truth tables to show the following sentences mean the same thing.

a) P iff Q means the same as $\sim P$ iff $\sim Q$
b) $\sim (P \Leftrightarrow Q)$ means the same as $(P \wedge \sim Q) \vee (\sim P \wedge Q)$
c) $P \Rightarrow Q$ means the same as $\sim P \vee Q$

6. Translate the given sentences in English to conditional form.

a) Unless you study you won't get a good grade.
b) Do you like it, it's yours.
c) Come here and I'll help you.
d) Get out or I'll call the cops.
e) Anyone who doesn't study deserves to flunk.
f) Criticize her and she will slap you.
g) With his toupee on the professor looks younger.

7. **(Distributive Laws for AND and OR)** For the sentences P, Q and R verify the **distributive laws**

a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

8. **(Inverse, Converse, and Contrapositive)** One of the following sentences has the same meaning as $P \Rightarrow Q$. Which one is it?

$$\textit{inverse}: \sim P \Rightarrow \sim Q$$
$$\textit{converse}: Q \Rightarrow P$$
$$\textit{contrapositive}: \sim Q \Rightarrow \sim P$$

For the two sentences which are not always true, give examples where they are true.

8. **(True or False?)** Which of the following conditional sentences are true?

   a) $(P \vee Q) \Rightarrow (P \wedge Q)$
   b) $(P \wedge Q) \Rightarrow (P \vee Q)$

9. Show the equivalence of the following implications.

   a) $P \Rightarrow Q$ (direct form of an implication)
   b) $\sim Q \Rightarrow \sim P$ (contrapositive form)
   c) $(P \wedge \sim Q) \Rightarrow \sim P$ (proof by contradiction)
   d) $(P \wedge \sim Q) \Rightarrow Q$ (proof by contradiction)
   e) $(P \wedge \sim Q) \Rightarrow R \wedge \sim R$ (proof by *reduction ad absurdum*)

10. Give, if possible, an example of a true conditional sentence for which

a) the contrapositive is true
b) the contrapositive is false
c) the converse is true
d) the converse is false

11. The **inverse** of the implication $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

a) Prove or disprove that an implication and its inverse are equivalent.
b) What are the truth values of P and Q for which an implication and its
c) inverse are both true?
d) What are the truth values of P and Q for which the implication and its inverse are both false?

12. For the sentence

"if N is a positive integer, then 2N is an even positive integer"

write the converse, contrapositive, and inverse sentences.

13. Let P,Q, and R be sentences. Show

a) $P \Rightarrow (Q \Leftrightarrow R)$ requires paranthesis

b) $(P \wedge Q) \vee R$ requires paranthesis

c) $(\sim P \vee Q) \Rightarrow R$ may be written $\sim P \vee Q \Rightarrow R$

15. **(Three-Valued Logic)** Two-valued (T and F) truth tables were basic in logic until 1921 when the Polish logician Jan Lukasiewicz (1878-1956) and American logician Emil Post (1897-1954) introduced n-valued logical systems where n is any integer greater than 1. For example, sentences in a three-valued logic might have values $T =$ true, $F =$ false, and $M =$ maybe. The truth table for $P \Rightarrow Q$ in this 3-valued system would be

P	Q	$P \Rightarrow Q$
T	T	**T**
T	M	**M**
T	F	**F**
M	T	**T**
M	M	**M**
M	F	**M**
F	T	**T**
F	M	**T**
F	F	**T**

16. (Modus Ponens and Modus Tollens) Modus Ponens[3] and Modus Tollens[4] are systematic ways of making logical arguments that takes the form

Modus Ponens
If P then Q
P
Therefore Q

Modus Tollens
If P then Q
$\sim Q$
Therefore $\sim P$

Show that Modus Tollens is simply the contrapositive of Modus Ponens.

17. Are the following two statements equivalent ?

[3] Latin: *mode that affirms.*

[4] Latin *mode that denies.*

$$P \wedge (Q \Rightarrow R)$$
$$(P \wedge Q) \Rightarrow R$$

17. **(Logical Puzzle)** On an isolated island the inhabitants either always tell the truth or always lie. You make a visit to the island and ask the first person you meet about their favorite baseball team. The person says two things:

- I like the Red Sox.
- If I like the Red Sox, then I like the Yankees.

Does the islander like the Red Sox? Does the islander like the Yankees? What is the reason for your answer?