MAT 563 HOMEWORK 1: DUE WEDNESDAY, JAN. 23

(1) Let G be a group with $|G| = 4$. Prove that G is abelian.

(2) Let A, B be subgroups of a group G. Prove or give a counterexample:
 (a) $A \cap B$ is a subgroup of G.
 (b) $A \cup B$ is a subgroup of G.

(3) Let G be a finite abelian group and $a, b \in G$. Let $n = |a|$ and $m = |b|$, and suppose $\gcd(n, m) = 1$. Prove that $|ab| = nm$. Does the conclusion still hold if G is nonabelian? Prove or give a counterexample.

(4) Let G be a group and let $a, b \in G$. Prove that $|ab| = |ba|$.

(5) Let p be a prime. Compute $|\operatorname{GL}_2(\mathbf{Z}/p\mathbf{Z})|$. (Hint: How many possible top rows are there? Given a top row, how many allowable bottom rows are there?) Generalize to find $|\operatorname{GL}_n(\mathbf{Z}/p\mathbf{Z})|$.

(6) Let S be a nonempty subset of a group G. The subgroup $\langle S \rangle$ generated by S is

$$\langle S \rangle = \{x_1 x_2 \cdots x_n \,|\, x_i \in S \text{ or } x_i^{-1} \in S\}.$$

Prove that

$$\langle S \rangle = \bigcap_{\substack{H < G, \\ S \subset H}} H.$$