

- (1) Let  $E$  be a splitting field over  $K$  of a separable degree  $n$  polynomial  $f(x) \in K[x]$ . Let  $\Sigma = \{a_1, \dots, a_n\} \subset E$  be the set of roots of  $f$ . Let  $G = \text{Gal}(E/K)$ . The group  $G$  has a natural action on the set  $\Sigma$  (since  $\sigma(a_i) = a_j$  for any  $\sigma \in G$ , as proven in class). Prove that the associated permutation representation

$$\chi : G \longrightarrow S_n$$

is one-to-one.

- (2) (Continue with same notation as above.) Suppose  $f$  is irreducible. Then as shown in class, the action of  $G$  on  $\Sigma$  is transitive. Suppose:
- $E \subset \mathbf{C}$
  - $\deg(f) = p$  is prime
  - $f$  has exactly two non-real roots.
- Prove that  $G \cong S_p$  is the full symmetric group. (Hint: recall that  $S_n$  is generated by a 2-cycle and an  $n$ -cycle.)

- (3) Let  $f(x) \in K[x]$  be an irreducible polynomial of degree 3, and let  $E$  be a splitting field. Define

$$\Delta = (a_1 - a_2)(a_1 - a_3)(a_2 - a_3) \in E,$$

where  $a_1, a_2, a_3 \in E$  are the roots of  $f$ . (See page 270 of Hungerford.)

(a) Show that  $\Delta^2 \in K$ . This number is the *discriminant* of  $f$ .

(b) Let  $G = \text{Gal}(E/K)$ . Note that by problem (1) above, we can identify  $G$  with a subgroup of  $S_3$  by its action on the roots

$$a_1, a_2, a_3. \text{ Show that } G = \begin{cases} A_3 & \text{if } \Delta \in K \\ S_3 & \text{otherwise.} \end{cases}$$

Remark: Given an arbitrary monic cubic polynomial

$$f(x) = x^3 + a_2x^2 + a_1x + a_0 \in K[x],$$

we can transform it to one in which  $a_2 = 0$  by:

$$f(x+c) = x^3 + ax + b$$

for  $c = -a_2/3$ . This does not affect the splitting field since the roots of  $f(x)$  and  $f(x+c)$  differ by  $-c \in K$ . Furthermore, because the discriminant involves only differences of the roots, this shows that  $f(x)$  and  $f(x+c)$  have the same discriminant. So without loss of generality we can take  $f(x) = x^3 + ax + b$ . Then a “gruesome” calculation (see Prop V.4.8) shows that

$$\Delta^2 = -(4a^3 + 27b^2) \in K.$$

- (4) Find an irreducible cubic polynomial in  $\mathbf{Q}[x]$  whose splitting field has Galois group  $S_3$ . Find another example where the Galois group is the cyclic group  $A_3$ . (Different from the ones given on page 272!)