MAT 563 HOMEWORK 2: Due Wednesday, Jan. 30

(1) Let G be a group of order 15. Show that G contains an element of order 3.

(2) Let G be a group, and let $N < G$ be a subgroup of index 2. Prove that N is normal in G.

(3) Determine all homomorphisms from $\mathbf{Z}$ to $\mathbf{Z}$.

(4) Let Z be the center of a group G (recall $Z \lhd G$). Prove that if the group G/Z is cyclic, then G is abelian.

(5) Let G be a group and let G' be the subgroup generated by the set

$$\{aba^{-1}b^{-1} \mid a, b \in G\}.$$

This is the *commutator subgroup* of G. It measures the degree to which G fails to be abelian. (E.g. $G' = \{e\}$ if G is abelian.)

(a) Prove that $G' \lhd G$ and that G/G' is abelian.

(b) If $H \lhd G$ and G/H is abelian, prove that $G' < H$. (Thus G/G' is the maximal abelian quotient of G.)