MAT 563 HOMEWORK 3: DUE WEDNESDAY, FEB. 6

(1) Let $G = \langle a \rangle$ be a cyclic group of order n. Use the first isomorphism theorem to prove that $G \cong \mathbf{Z}/n\mathbf{Z}$.

(2) Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, c \in \mathbf{R}, ad \neq 0 \right\}$.

(a) Show that B is a subgroup of $\mathrm{GL}_2(\mathbf{R})$, and that B is nonabelian.

(b) Let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbf{R} \right\}$, called the **unipotent subgroup** of $\mathrm{GL}_2(\mathbf{R})$. Prove that $B/N \cong \mathbf{R}^* \times \mathbf{R}^*$, where $\mathbf{R}^* \times \mathbf{R}^*$ is the set $\{(x, y) | x, y \in \mathbf{R}^*\}$ with the abelian group operation $(x, y) \cdot (a, b) = (xa, yb)$. (Use the 1st isomorphism theorem.)

(c) Show that N is the commutator subgroup of B. (By part (b) you know that N contains it.)

(3) In S_n, an **elementary 2-cycle** is a 2-cycle of the form $(a \ \ a{+}1)$.

(a) Prove that every 2-cycle is the product of an odd number of elementary 2-cycles. Do not use Theorem 6.7 since we will use the result of this problem to prove that theorem.

(b) Conclude that S_n is generated by the elementary 2-cycles.

(c) Show that S_n is also generated by the two elements $(1\ 2)$ and $(1\ 2\ \ldots\ n)$.

(4) In a group G, the **conjugacy class** of an element $a \in G$ is the set

$$K(a) = \{gag^{-1} \mid g \in G\}.$$

A **conjugate** of a is an elment of $K(a)$, i.e. any elment of the form gag^{-1}. In this exercise you will investigate conjugacy classes in S_n.

(a) Show that a conjugate of a k-cycle is again a k-cycle: Choose any k-cycle $\phi \in S_n$. Then

$$\phi = (a_1 \ \ a_2 \ \ldots \ a_k),$$

for some distinct $a_i \in \{1, 2, \ldots, n\}$. Let $\tau \in S_n$ be any permutation. Compute $\tau\phi\tau^{-1}$. (Hint: where does $\tau\phi\tau^{-1}$ send $\tau(a_1)$?)

(b) Let ϕ be a k-cycle as above. Show that

$$K(\phi) = \{k\text{−cycles} \in S_n\}.$$

(You showed $\subset$ in part (a). You just have to show $\supset$.)

(5) Consider a tetrahedron T (a pyramid with 4 equal triangular sides). Label the vertices $1, 2, 3, 4$. Find the symmetry group of T, i.e. the subgroup of S_4 consisting of permutations which give rotations of T.