MAT 563 HOMEWORK 4: DUE WEDNESDAY, FEB. 13

(1) Let G be a group, and let $D = \{(x, x) | \, x \in G\}$ be a subset of $G \times G$.

(a) Show that $D < G \times G$.

(b) Prove that $D \lhd G \times G$ if and only if G is abelian.

(c) Suppose G is abelian. What is $(G \times G)/D$ isomorphic to?

(2) Let $H \lhd G_1$ and $K \lhd G_2$. Show that $G_1 \times G_2/(H \times K) \cong G_1/H \times G_2/K$.

(3) Let G be a nonabelian group, and let $X = G$. Show that

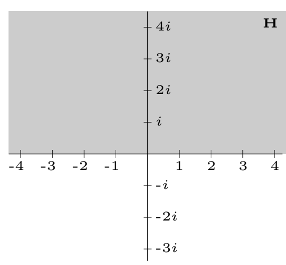
$$g \cdot x = xg$$

does not give a group action, in other words, G does not act on itself by right multiplication. Show that

$$g \cdot x = xg^{-1}$$

does define a group action.

(4) Let $G = \mathrm{SL}_2(\mathbf{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | \, ad - bc = 1\}$, and let $\mathbf{H} = \{x + iy | \, y > 0\}$ be the complex upper half-plane:



For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathbf{H}$, define $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$.

(a) Prove that this defines a group action of G on $\mathbf{H}$. You need to check three things: that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z$ actually belongs to $\mathbf{H}$, and that this operation satisfies the two conditions in the definition of group action.

(b) Prove that the action is transitive. (Hint: show that the orbit of i is all of $\mathbf{H}$.)

(c) Prove that the stabilizer of $i \in \mathbf{H}$ is the subgroup

$$\mathrm{SO}(2) = \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} | \theta \in \mathbf{R}\} < G.$$

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(5) Let $H < G$, and let G act on G/H by left translation. Let $\chi : G \longrightarrow \mathcal{A}(G/H)$ be the resulting permutation representation of G.

(a) Determine the kernel K of χ.

(b) Show that $K < H$.

(c) Show that if $N \lhd G$ and $N < H$, then $N < K$.

(6) Let G be a finite group, and let $H < G$ be a subgroup of prime order p. Suppose $|G| = pn$ with $p > n$. Prove that H is normal in G. (Comments: This is problem 14 on page 92. The idea is use problem (5) above, as in Corollary 4.10.)