MAT 563 HOMEWORK 8: Due Wednesday, Mar. 26

(1) Let R be a finite integral domain. Prove that R is a field.

(2) A ring R has *characteristic* n if n is the least positive integer such that $nr = 0$ for all $r \in R$. If no such integer exists (as is the case for $\mathbf{Z}$), then R has characteristic 0. Suppose R has characteristic p, where p is prime. Prove the "Freshman's dream":

$$(a+b)^{p^k} = a^{p^k} + b^{p^k}$$

for all $a, b \in \mathbf{R}$ and $k \geq 0$.

(3) Let R be a commutative ring with 1, and of characteristic p. Show that the map $\phi : R \longrightarrow R$ defined by $\phi(r) = r^p$ is a ring homomorphism. (ϕ is called the *Frobenius map.*)

(4) Let $\phi : R \longrightarrow S$ be a ring homomorphism, where R, S are commutative rings with 1. Prove or give a counterexample: $\phi(1_R) = 1_S$.

(5) Let $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ be an automorphism of the field of real numbers. Suppose ϕ is continuous. Prove that ϕ is the identity map. Now show that in fact *every* automorphism of $\mathbf{R}$ is continuous, and conclude that $\mathbf{R}$ has no nontrivial automorphisms.

(6) Let I be an ideal of a commutative ring R. Define

$$Rad(I) = \{r \in R | \, r^n \in I \text{ for some } n\}.$$

Prove that $Rad(I)$ is an ideal (called the *radical* of I).